SIMILARITY VERSUS COINCIDENCE ROTATIONS OF LATTICES

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Abstract. The groups of similarity and coincidence rotations of an arbitrary lattice Γ in d-dimensional Euclidean space are considered. It is shown that the group of similarity rotations contains the coincidence rotations as a normal subgroup. Furthermore, the structure of the corresponding factor group is examined. If the dimension d is a prime number, this factor group is an elementary Abelian d-group. Moreover, if Γ is a rational lattice, the factor group is trivial (d odd) or an elementary Abelian 2-group (d even).

1. Introduction

The classification of colour symmetries and that of grain boundaries in crystals and quasicrystals are intimately related to the existence of similar and coincidence sublattices of the underlying lattice of periods or the corresponding translation module. It is thus of interest to understand the corresponding groups of isometries from a more mathematical perspective. An example for the structure of the groups of coincidence rotations and similarity rotations of planar lattices is considered and the factor group of similarity modulo coincidence rotations is calculated. More generally, for lattices in d dimensions, we show that the factor group is the direct sum of cyclic groups of prime power orders that divide d. In the case of rational lattices, which include hypercubic lattices and all root lattices, this means that the factor group is either trivial or an elementary Abelian 2-group, depending on the parity of d.

2. Coincidence Rotations

A *lattice* in $\mathbb{R}^d$ is a subgroup of the form

$$\Gamma = \mathbb{Z}b_1 \oplus \mathbb{Z}b_2 \oplus \ldots \oplus \mathbb{Z}b_d,$$

where $\{b_1, \ldots, b_d\}$ is a basis of $\mathbb{R}^d$. Two lattices Γ, Γ' in $\mathbb{R}^d$ are called *commensurate* if their intersection $\Gamma \cap \Gamma'$ has finite index both in Γ and in Γ'. In this case, we write $\Gamma \sim \Gamma'$. Commensurateness of lattices is an equivalence relation (cf. [1]). An element $R \in \mathrm{SO}(d)$ is called a *coincidence rotation of* Γ, if $\Gamma \sim R\Gamma$. We thus define

$$\mathrm{SOC}(\Gamma) := \{\, R \in \mathrm{SO}(d) \mid \Gamma \sim R\Gamma \,\},$$

which is a subgroup of $\mathrm{SO}(d)$.

Example 2.1 (The square lattice $\mathbb{Z}^2$)**.** As shown in Thm. 3.1 of [1], the coincidence rotations of $\mathbb{Z}^2$ are precisely the special orthogonal matrices with rational entries,

$$\mathrm{SOC}(\mathbb{Z}^2) = \mathrm{SO}(2, \mathbb{Q}).$$

On the other hand, one can identify $\mathbb{Z}^2$ with the Gaussian integers $\mathbb{Z}[i]$, where i is the imaginary unit. Then, a rotation $R(\varphi)$ with rotation angle φ corresponds to a multiplication with the

complex number $e^{i\varphi} \in \bigl(\mathbb{Q}(i) \cap \mathbb{S}^1\bigr) \simeq \mathrm{SOC}(\mathbb{Z}^2)$; see [5]. Using the fact that $\mathbb{Z}[i]$ is a unique factorisation domain, each coincidence rotation uniquely factorises as

$$e^{i\varphi} = \varepsilon \prod_{p \equiv 1(4)} \left(\frac{\omega_p}{\overline{\omega_p}} \right)^{n_p}, \tag{1}$$

where ε is a unit in $\mathbb{Z}[i]$, $n_p \in \mathbb{Z}$ with only finitely many of them nonzero, p runs through the rational primes congruent to 1 (mod 4), and p factorises as $p = \omega_p \overline{\omega_p}$ in $\mathbb{Z}[i]$ with $\omega_p / \overline{\omega_p}$ not a unit. This shows that $\mathrm{SOC}(\mathbb{Z}^2)$ is a countably generated Abelian group. More precisely,

$$\mathrm{SOC}(\mathbb{Z}^2) = C_4 \times \mathbb{Z}^{(\aleph_0)},$$

where C_4 denotes the cyclic group of order 4 (here generated by i) and $\mathbb{Z}^{(\aleph_0)}$ stands for the direct sum of countably many infinite cyclic groups, which are here generated by the $\omega_p / \overline{\omega_p}$ with $p \equiv 1 \pmod 4$ (cf. [5]).

3. Similarity Rotations

Let $\varGamma \subset \mathbb{R}^d$ again be a lattice. Define

$$\mathrm{SOS}(\varGamma) := \{\, R \in \mathrm{SO}(d) \mid \varGamma \sim \alpha R \varGamma \ \text{ for some } \alpha > 0 \,\}.$$

The elements of $\mathrm{SOS}(\varGamma)$ are called *similarity rotations*. $\mathrm{SOS}(\varGamma)$ is a group (cf. [4]) and contains $\mathrm{SOC}(\varGamma)$ as a subgroup.

Example 3.1 ($\mathbb{Z}^2$)**.** For $\mathbb{Z}^2$, the group of similarity rotations consists precisely of the set of $\mathbb{Z}^2$-directions,

$$\mathrm{SOS}(\mathbb{Z}^2) = \left\{ \frac{a}{|a|} \,\middle|\, 0 \neq a \in \mathbb{Z}[i] \right\}. \tag{2}$$

We parametrise the Euclidean plane by the complex numbers $\mathbb{C}$, and use $\mathrm{SO}(2) \simeq \mathbb{S}^1$ and $\mathbb{Z}^2 = \mathbb{Z}[i]$. To show (2), let $z \in \mathbb{Z}[i] \setminus \{0\}$. Since $\mathbb{Z}[i]$ is a ring, one has

$$|z| \cdot \frac{z}{|z|}\, \mathbb{Z}[i] \subset \mathbb{Z}[i],$$

so that $z/|z| \in \mathrm{SOS}(\mathbb{Z}^2)$. Conversely, let $r \in \mathrm{SOS}(\mathbb{Z}^2)$, meaning that $r \in \mathbb{S}^1$ with $\lambda r \mathbb{Z}[i] \sim \mathbb{Z}[i]$ for some $\lambda > 0$. By Remark 4.2 below, there exists a nonzero integer t with $t\lambda r \mathbb{Z}[i] \subset \mathbb{Z}[i]$. Since $1 \in \mathbb{Z}[i]$, this yields $t\lambda r \in \mathbb{Z}[i]$, say $t\lambda r = v$. Thus $|t\lambda| = |v|$, because $r \in \mathbb{S}^1$. This shows that $r = v/|v|$ is a $\mathbb{Z}[i]$-direction.

Each nonzero element of $\mathrm{SOS}(\mathbb{Z}^2)$ is thus of the form $z/|z|$ with $0 \neq z \in \mathbb{Z}[i]$. Using unique factorisation in $\mathbb{Z}[i]$ again, we get

$$\frac{z}{|z|} = \left(\frac{1+i}{\sqrt{2}} \right)^k \prod_{p \equiv 1(4)} \left(\frac{\omega_p}{\sqrt{p}} \right)^{\ell_p},$$

where $0 \leqslant k < 8$ and $\ell_p \in \mathbb{Z}$ (other restrictions as in (1)). One observes that $(1+i)/\sqrt{2}$ is a primitive 8th root of unity, hence it generates the cyclic group C_8. Furthermore, one finds

$$\left(\frac{\omega_p}{\sqrt{p}} \right)^2 = \frac{\omega_p^2}{\omega_p \overline{\omega_p}} = \frac{\omega_p}{\overline{\omega_p}}.$$

This shows that the generators of $\mathrm{SOC}(\mathbb{Z}^2) = C_4 \times \mathbb{Z}^{(\aleph_0)}$ are the squares of the generators of $\mathrm{SOS}(\mathbb{Z}^2)$. Thus

$$\mathrm{SOC}(\mathbb{Z}^2) = \{ x^2 \mid x \in \mathrm{SOS}(\mathbb{Z}^2) \} =: \big(\mathrm{SOS}(\mathbb{Z}^2)\big)^2 .$$

The following more general result was shown in [5]: For all cyclotomic fields $\mathbb{Q}(\xi_n)$ of class number one (excluding $\mathbb{Q}$), one has

$$\mathrm{SOC}(\mathcal{O}_n) \simeq C_{N(n)} \times \mathbb{Z}^{(\aleph_0)},$$

where $\mathcal{O}_n = \mathbb{Z}[\xi_n]$ is the ring of integers in $\mathbb{Q}(\xi_n)$ and $N(n) = \mathrm{lcm}(n, 2)$.

Returning to our example, we find the structure of the factor group to be

$$\begin{aligned} \mathrm{SOS}(\mathbb{Z}^2)/\,\mathrm{SOC}(\mathbb{Z}^2) &\simeq (C_8/C_4) \times C_2^{(\aleph_0)} \\ &\simeq C_2 \times C_2^{(\aleph_0)}, \end{aligned}$$

where $C_2^{(\aleph_0)}$ stands for the direct sum of countably many cyclic groups of order 2. Hence, the factor group is the direct sum of cyclic groups of order 2, which means that it is an elementary Abelian 2-group. More generally, for arbitrary lattices in Euclidean d-space, we shall see below that the group SOC is a normal subgroup of SOS, whence the factor group always exists.

4. Factor Group

Throughout this section, let Γ be a lattice in $\mathbb{R}^d$, with $d \geq 2$.

Definition 4.1. For an arbitrary element $R \in \mathrm{SO}(d)$, define

$$\mathrm{scal}_\Gamma(R) = \{ \alpha \in \mathbb{R} \mid \Gamma \sim \alpha R \Gamma \}.$$

Note that

$$\mathrm{SOS}(\Gamma) = \{ R \in \mathrm{SO}(d) \mid \mathrm{scal}_\Gamma(R) \neq \varnothing \}.$$

Remark 4.2. If $\alpha \in \mathrm{scal}_\Gamma(R)$, then there exists a nonzero integer t such that $t\alpha R\Gamma \subset \Gamma$. Namely, if $\alpha \in \mathrm{scal}_\Gamma(R)$, the group index $[\alpha R\Gamma : (\Gamma \cap \alpha R\Gamma)] = t$ is finite. Consequently, one has $t\alpha R\Gamma \subset (\Gamma \cap \alpha R\Gamma) \subset \Gamma$.

Lemma 4.3. *For $R \in \mathrm{SOS}(\Gamma)$, the following assertions hold.*

1. $b \cdot \mathrm{scal}_\Gamma(R) \subset \mathrm{scal}_\Gamma(R)$ *for all* $b \in \mathbb{Q} \setminus \{0\}$
2. $r\Gamma \sim \Gamma$ *with* $r \in \mathbb{R}$ *implies* $r \in \mathbb{Q}$
3. $\alpha\beta^{-1} \in \mathbb{Q}$ *for all* $\alpha, \beta \in \mathrm{scal}_\Gamma(R)$

Proof. Let $\alpha \in \mathrm{scal}_\Gamma(R)$. For $b = b_1/b_2$ with $b_1, b_2 \in \mathbb{Z} \setminus \{0\}$, one finds

$$\frac{b_1}{b_2}\alpha R\Gamma \sim \frac{1}{b_2}\alpha R\Gamma \sim \frac{1}{b_2}\Gamma \sim \Gamma.$$

This proves (1). In order to show (2), let $r \in \mathbb{R}$ with $r\Gamma \sim \Gamma$. By Remark 4.2, there exists a nonzero integer k with $kr\Gamma \subset \Gamma$. Now, let $\gamma \in \Gamma$ be represented in terms of a basis $\{\gamma_1, \ldots, \gamma_d\}$ of Γ as $\gamma = \sum_{i=1}^d c_i\gamma_i$, with $c_i \in \mathbb{Z}$. On the other hand, $kr\gamma$ can be represented as $kr\gamma = \sum_{i=1}^d a_i\gamma_i$, where $a_i \in \mathbb{Z}$. Thus

$$\sum_{i=1}^d krc_i\gamma_i = \sum_{i=1}^d a_i\gamma_i.$$

By assumption, $\varGamma$ spans $\mathbb{R}^d$, so that $\{\gamma_1, \dots, \gamma_d\}$ forms an $\mathbb{R}$-basis of $\mathbb{R}^d$. Therefore, one has $krc_i = a_i$, yielding $r = a_i c_i^{-1} k^{-1} \in \mathbb{Q}$. Finally, (3) is obtained from (2) as follows. By assumption, one has

$$\beta R\varGamma \sim \varGamma \sim \alpha R\varGamma.$$

Multiplying with $1/\beta$ gives $R\varGamma \sim \frac{\alpha}{\beta} R\varGamma$, which completes the proof. □

Denote by $\mathbb{R}^\bullet$ (by $\mathbb{Q}^\bullet$) the multiplicative groups formed by the nonzero real (rational) numbers. Define a map

$$\eta \colon \mathrm{SOS}(\varGamma) \longrightarrow \mathbb{R}^\bullet / \mathbb{Q}^\bullet$$

by

$$R \longmapsto [\alpha],$$

where $[\,\cdot\,]$ denotes the equivalence classes of $\mathbb{R}^\bullet/\mathbb{Q}^\bullet$ and α is an arbitrary element of $\mathrm{scal}_\varGamma(R)$. This map is well-defined due to the fact that $\mathrm{scal}_\varGamma(R)$ is non-empty for $R \in \mathrm{SOS}(\varGamma)$ and by Lemma 4.3(3).

Lemma 4.4. *The map η is a group homomorphism with* $\mathrm{Ker}(\eta) = \mathrm{SOC}(\varGamma)$.

Proof. Let $R, S \in \mathrm{SOS}(\varGamma)$ and choose $\alpha \in \mathrm{scal}_\varGamma(R)$ and $\beta \in \mathrm{scal}_\varGamma(S)$. We need to show that $\alpha\beta \in \mathrm{scal}_\varGamma(RS)$. By assumption, one has

$$\varGamma \sim \alpha R\varGamma \sim \alpha R(\beta S\varGamma) = \alpha\beta RS\varGamma.$$

Thus $\alpha\beta \in \mathrm{scal}_\varGamma(RS)$, hence η is a group homomorphism. It remains to show that $\mathrm{Ker}(\eta) = \mathrm{SOC}(\varGamma)$. For $R \in \mathrm{SOC}(\varGamma)$, the set $\mathrm{scal}_\varGamma(R)$ contains 1, which means $R \in \mathrm{Ker}(\eta)$. Conversely, if $S \in \mathrm{Ker}(\eta)$, one has $\mathrm{scal}_\varGamma(S) \subset \mathbb{Q}$. Let $\mu \in \mathrm{scal}_\varGamma(S)$. Due to Lemma 4.3(1), we have $1 = \mu^{-1}\mu \in \mathrm{scal}_\varGamma(S)$, which proves $S \in \mathrm{SOC}(\varGamma)$. □

Since $\mathrm{SOC}(\varGamma)$ is the kernel of a group homomorphism, it is a normal subgroup of $\mathrm{SOS}(\varGamma)$, so that the factor group $\mathrm{SOS}(\varGamma) \,/\, \mathrm{SOC}(\varGamma)$ can be considered. It is isomorphic to the image of η, which is a subgroup of $\mathbb{R}^\bullet/\mathbb{Q}^\bullet$ and thus Abelian. To examine the structure of the factor group $\mathrm{SOS}(\varGamma) \,/\, \mathrm{SOC}(\varGamma)$, we need the following result from the theory of Abelian groups.

Theorem 4.5. *Let G be a countable Abelian group.*

(1) *If a prime number p exists such that $x^p = 1$ for all $x \in G$, then G is the direct sum of subgroups of order p.*
(2) *If a positive integer n exists such that $x^n = 1$ for all $x \in G$, then G is the direct sum of cyclic groups of prime power orders that divide n.*

Proof. See [6, Thms. 5.1.9 and 5.1.12]. □

Remark 4.6. Let $R \in \mathrm{SOS}(\varGamma)$. For all elements $\alpha \in \mathbb{R}$ with $\alpha R\varGamma \subset \varGamma$, one has $|\alpha^d| = [\varGamma : \alpha R\varGamma] \in \mathbb{N}$. This follows via the determinants of basis matrices of the lattices involved. Consequently, α is an algebraic number.

Theorem 4.7. *The group* $\mathrm{SOS}(\varGamma) \,/\, \mathrm{SOC}(\varGamma)$ *is countable. Furthermore, it is the direct sum of cyclic groups of prime power orders that divide d.*

Proof. We consider again the group homomorphism $\eta \colon \mathrm{SOS}(\varGamma) \longrightarrow \mathbb{R}^\bullet/\mathbb{Q}^\bullet$. Let $R \in \mathrm{SOS}(\varGamma)$. This implies $\eta(R) = [\alpha]$ for some element $\alpha \in \mathrm{scal}_\varGamma(R)$. Due to Remark 4.2, there exists a nonzero integer t with $t\alpha R\varGamma \subset \varGamma$. Furthermore, one has $\eta(R) = [t\alpha]$. By Remark 4.6, $t\alpha$ is an algebraic number. This means that all elements of $\eta(\mathrm{SOS}(\varGamma))$ are represented by

algebraic numbers. Thus, since the set of algebraic numbers is countable, also the group $\mathrm{SOS}(\Gamma) \,/\, \mathrm{SOC}(\Gamma)$ is countable.

According to Remark 4.6, one has $(t\alpha)^d \in \mathbb{Q}$, which yields

$$\eta(R)^d = [t\alpha]^d = [(t\alpha)^d] = [1] \tag{3}$$

in $\mathbb{R}^\bullet/\mathbb{Q}^\bullet$. Using the group isomorphism $\eta(\mathrm{SOS}(\Gamma)) \simeq \mathrm{SOS}(\Gamma) \,/\, \mathrm{SOC}(\Gamma)$, this shows that the order of each element of $\mathrm{SOS}(\Gamma) \,/\, \mathrm{SOC}(\Gamma)$ divides d. Theorem 4.5(2) then implies that the group $\mathrm{SOS}(\Gamma) \,/\, \mathrm{SOC}(\Gamma)$ is the direct sum of cyclic groups of prime power orders. Consequently, the prime power order of each cyclic group divides d. □

Corollary 4.8. *If $d = p$ is a prime number, the factor group $\mathrm{SOS}(\Gamma) \,/\, \mathrm{SOC}(\Gamma)$ is an elementary Abelian p-group, i.e., it is the direct sum of cyclic groups of order p.* □

Corollary 4.9 (Rational Lattices)**.** *Let Γ be a lattice in $\mathbb{R}^d$ such that $\langle x, x\rangle \in \mathbb{Q}$ for all $x \in \Gamma$, where $\langle \cdot\, , \cdot \rangle$ denotes the standard scalar product in $\mathbb{R}^d$. Lattices satisfying the above property are also called* rational (cf. [2])*. For these lattices, the group $\mathrm{SOS}(\Gamma) \,/\, \mathrm{SOC}(\Gamma)$ is an elementary Abelian 2-group when d is even. If d is odd, one has $\mathrm{SOS}(\Gamma) = \mathrm{SOC}(\Gamma)$. Either way, one has*

$$(\mathrm{SOS}(\Gamma))^2 \subset \mathrm{SOC}(\Gamma)\,.$$

Proof. Let $R \in \mathrm{SOS}(\Gamma)$. By Remark 4.2, there exists a nonzero real number α such that $\alpha R\Gamma \subset \Gamma$. By assumption, one has $\langle \alpha R\gamma, \alpha R\gamma\rangle \in \mathbb{Q}$ for all $\gamma \in \Gamma$. Hence $\alpha^2 \in \mathbb{Q}$, say $\alpha^2 = r/s$, where $r, s \in \mathbb{Z} \setminus \{0\}$. Since $s\alpha^2 = r \in \mathbb{Z}$ and $\alpha R\Gamma \subset \Gamma$, one gets

$$\Gamma \supset s\alpha R(\alpha R\Gamma) = s\alpha^2 R^2\Gamma \subset R^2\Gamma,$$

whence

$$rR^2\Gamma \subset \big(\Gamma \cap R^2\Gamma\big)\,.$$

Thus both $[\Gamma : rR^2\Gamma]$ and $[R^2\Gamma : rR^2\Gamma]$ are finite. This implies $\Gamma \sim R^2\Gamma$, so that R^2 is a coincidence rotation of Γ. Consequently, $(\mathrm{SOS}(\Gamma))^2 \subset \mathrm{SOC}(\Gamma)$. This means that every element of the factor group $\mathrm{SOS}(\Gamma) \,/\, \mathrm{SOC}(\Gamma)$ is of order 1 or 2. Thus, the factor group is an elementary Abelian 2-group by Theorem 4.5(1).

If d is odd, set $d = 2m + 1$ with $m \in \mathbb{N}$. Then

$$\alpha(\alpha^2)^m = \alpha^d \;\in \mathbb{Q}$$

yields $\alpha \in \mathbb{Q}$, because $\alpha^2 \in \mathbb{Q}$. Thus $\eta(R) = [\alpha] = [1]$ in $\mathbb{R}^\bullet/\mathbb{Q}^\bullet$ for all $R \in \mathrm{SOS}(\Gamma)$, whence $\mathrm{SOS}(\Gamma) \,/\, \mathrm{SOC}(\Gamma)$ is the trivial group. □

5. Outlook

In view of Penrose tilings and similar models, where the translation module is not a lattice, it is desirable to generalise the above notions of similarity and coincidence rotations from lattices to modules. Some progress has been made in this direction for certain modules over subrings S of the rings of integers of real algebraic number fields. More precisely, similar results [3] to those presented here hold for S-modules of rank d that span $\mathbb{R}^d$.

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